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Note on the Faxén relations for a particle in Stokes flow

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The Faxén relations for a rigid particle in an arbitrary Stokes flow are generalized to give expressions for the stresslet (and higher stress moments) exerted by the particle on the fluid, and also to viscous drops immersed in a viscous fluid.

1. Introduction

Faxén (1924) considered the problem of determining the force $\mathbf{F}$ and couple $\mathbf{L}$ exerted by an arbitrary Stokes flow on a stationary rigid sphere in an unbounded fluid at zero Reynolds number. He showed that in a fluid of viscosity μ, $\mathbf{F}$ and $\mathbf{L}$ are given in terms of the prescribed fluid flow at infinity $\mathbf{u}_\infty(\mathbf{r})$ by

$$\mathbf{F} = 6\pi\mu a\,(\mathbf{u}_\infty + \tfrac{1}{6}a^2\nabla^2\mathbf{u}_\infty)^0, \tag{1}$$

$$\mathbf{L} = 8\pi\mu a^3 \times \tfrac{1}{2}(\nabla\wedge\mathbf{u}_\infty)^0, \tag{2}$$

where the superscript zero indicates evaluation at the sphere centre and a is the sphere radius. These relations are known as Faxén's first and second laws. A similar result was obtained by Batchelor & Green (1972) in computing the stress in a dilute suspension of rigid spheres. They demonstrated that the stresslet $\mathbf{S}$ exerted by an isolated sphere on the fluid is

$$\mathbf{S} = \tfrac{20}{3}\,\pi\mu a^3 \times \tfrac{1}{2}[\nabla\mathbf{u}_\infty + (\nabla\mathbf{u}_\infty)^{\mathrm{T}} + \tfrac{1}{10}a^2\nabla^2(\nabla\mathbf{u}_\infty + (\nabla\mathbf{u}_\infty)^{\mathrm{T}})]^0. \tag{3}$$

There are two directions in which these results might usefully be extended. The first is to higher moments of the stress distribution and the second to particles other than rigid spheres. The computation of higher stress moments is precisely what is required to carry out a reflexions expansion for the interaction of two or more particles in Stokes flow: in the evaluation of hydrodynamic forces on the particles correct to $O(l/R)^n$, with R the particle separation and l the largest particle length, a knowledge of the $(n-1)$th stress moment for each particle in unbounded fluid is both necessary and sufficient. The details are given in Rallison (1977).

The generalization of the first and second Faxén laws to *rigid* particles of other shapes was made by Brenner (1964). For a rigid ellipsoid of semi-axes (a, b, c) in the directions of unit vectors $(\mathbf{p}, \mathbf{q}, \mathbf{r})$, for instance, he showed that

$$\mathbf{F} = \tfrac{4}{3}\pi abc\mu\mathbf{X}.\left\{\left(1+\frac{1}{3!}D^2+\frac{1}{5!}D^4+\dots\right)\mathbf{u}_\infty\right\}^0,$$

$$\mathbf{L} = \tfrac{4}{3}\pi abc\mu\mathbf{Y}.\left\{\left(1+\frac{2\cdot 3!}{5!}D^2+\frac{3\cdot 3!}{7!}D^4+\dots\right)\nabla\wedge\mathbf{u}_\infty\right\}^0,$$

where $$D^2 \equiv a^2\partial^2/\partial x^2 + b^2\partial^2/\partial y^2 + c^2\partial^2/\partial z^2$$

and the tensors $\mathbf{X}$ and $\mathbf{Y}$ are given in terms of elliptic integrals. (The result for $\mathbf{L}$ given here is simpler than Brenner's but equivalent to it.) Brenner also made the important simplifying observation that, by means of the reciprocal theorem, the problem of determining the force or couple on a particle in an arbitrary flow may be reduced to that of finding the stress induced by the same particle in translation or rotation respectively. For the latter problems the solution is frequently known, and at any rate is easier to determine than that for the fully general velocity field (though the simplification has not always been exploited, e.g. see Chwang 1975).

The purpose of this note is first to demonstrate that Faxén relations for the higher moments of the stress (in particular the stresslet) can be generated by methods analogous to Brenner's, and second to derive the corresponding results for viscous drops rather than rigid particles. By way of illustration of the techniques, we calculate the stresslet due to a rigid ellipsoid and also the force on and stresslet due to a viscous drop held almost spherical by surface tension.

2. Rigid particles

2.1. *General formulation*

On subtracting the prescribed flow at infinity, the fluid velocity $\mathbf{u}$ outside the particle satisfies

$$\left.\begin{aligned}\nabla.\mathbf{u} &= 0,\\ \nabla.\boldsymbol{\sigma} &= 0, \quad \boldsymbol{\sigma} = -p\mathbf{I}+\mu(\nabla\mathbf{u}+(\nabla\mathbf{u})^{\mathrm{T}}),\end{aligned}\right\} \tag{4}$$

with $\mathbf{u}\to 0$ as $r\to\infty$ and $\mathbf{u} = -\mathbf{u}_\infty$ on S, the particle surface. In addition, $\mathbf{u}_\infty$ and the corresponding stress $\boldsymbol{\sigma}_\infty$ are assumed to satisfy the Stokes equations (4).

The fundamental result which we exploit is the *reciprocal theorem* (Happel & Brenner 1965, § 3): if $(\mathbf{u}, \boldsymbol{\sigma})$ and $(\mathbf{u}', \boldsymbol{\sigma}')$ are two different solutions of (4) corresponding respectively to boundary conditions $\mathbf{u} = -\mathbf{u}_\infty$ and $\mathbf{u}' = -\mathbf{u}'_\infty$ on S, then

$$\int_S \mathbf{u}'_\infty.\boldsymbol{\sigma}.\mathbf{n}\,dS = \int_S \mathbf{u}_\infty.\boldsymbol{\sigma}'.\mathbf{n}\,dS,$$

or in a form more suitable for our purposes,

$$\int_S (\mathbf{u}'_\infty.\boldsymbol{\sigma}.\mathbf{n}-\mathbf{u}.\boldsymbol{\sigma}'_\infty.\mathbf{n})\,dS = \int_S \mathbf{u}_\infty.(\boldsymbol{\sigma}'+\boldsymbol{\sigma}'_\infty).\mathbf{n}\,dS. \tag{5}$$

Now the choice of different 'primed' fields generates a sequence of identities for the stress and its moments. In particular, choosing $\mathbf{u}'_\infty = \mathbf{U}'$ reduces the left-hand side to $\mathbf{U}'.\mathbf{F}$, while $\mathbf{u}'_\infty = \boldsymbol{\Omega}'\wedge\mathbf{r}$ gives $\boldsymbol{\Omega}'.\mathbf{L}$, and $\mathbf{u}'_\infty = \mathbf{E}'.\mathbf{r}$ gives $\mathbf{E}':\mathbf{S}$. The stresses $\boldsymbol{\sigma}'+\boldsymbol{\sigma}'_\infty$ on the right-hand side are then (known) linear functions of the forcings $\mathbf{U}'$, $\boldsymbol{\Omega}'$ and $\mathbf{E}'$, and as these tensors are arbitrary, an expression for $\mathbf{F}$, $\mathbf{L}$ or $\mathbf{S}$ is obtained by 'cancellation' of the primed quantities. There are plainly such identities at all orders.

Brenner (1963) suggested a convenient scheme for displaying the results for the force and couple on a particle translating and rotating, namely the use of a *matrix* of resistance tensors. Hinch (1972) generalized this scheme to include the stresslet. We see that, within the same formalism, results may be given for all the stress moments in terms of velocity gradients of all orders. In addition it may be shown (Rallison 1977) that this 'grand resistance matrix' of tensor coefficients is symmetric and positive definite, natural extensions of the results of Brenner (1963) and Hinch (1972).

The single most important quantity for the calculation of suspension rheology is the *stresslet*, and we illustrate the ideas of this section by computing the stresslet for an ellipsoid.

2.2. *The stresslet exerted by an ellipsoid in a general flow*

As described above, in order to determine $\mathbf{S}$ via the reciprocal theorem, we choose the (energy dissipation rate) conjugate field $\mathbf{u}'_\infty = \mathbf{E}'.\mathbf{r}$. The problem of determining $\mathbf{u}'$ was solved by Jeffery (1922) and, as noted by Cerf (1951), has the property that

$$\boldsymbol{\sigma}'(\mathbf{r}).\mathbf{n}(\mathbf{r})|_S = \mathbf{T}'.\mathbf{n}(\mathbf{r}),$$

where $\mathbf{T}'$ is a *constant* (proportional to $\mathbf{E}'$). It follows from (5) that

$$\mathbf{E}':\mathbf{S} = \mu\mathbf{E}':\mathbf{Z}:\int_S \mathbf{u}_\infty \mathbf{n}\, dS,$$

where the Jeffery solution shows that (referred to the natural axes $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ for the ellipsoid)

$$Z_{\alpha\beta\gamma\delta} = \frac{4J_1(p_\alpha p_\beta - \frac{1}{3}\delta_{\alpha\beta})(p_\gamma p_\delta - \frac{1}{3}\delta_{\gamma\delta})}{J_1J_2 + J_2J_3 + J_3J_1} + \left\{\frac{2}{I_1} + \frac{1}{4}\frac{(b^2-c^2)^2}{J_1(b^2+c^2) + 2I_1b^2c^2}\right\}$$
$$\times (q_\alpha r_\beta + q_\beta r_\alpha)\,(q_\gamma r_\delta + q_\delta r_\gamma) + \ldots$$

and the elliptic integrals I_i and J_i are given by Batchelor (1970). Now, with the same notation as in § 1,

$$\mathbf{S} = \mu\mathbf{Z}:\int_S \mathbf{u}_\infty \mathbf{n}\, dS = \mu\mathbf{Z}:\int_V \nabla\mathbf{u}_\infty\, dV$$
$$= \tfrac{4}{3}\pi abc\mu\mathbf{Z}:\left\{\left(1 + \frac{2\,.\,3!}{5!}D^2 + \frac{3\,.\,3!}{7!}D^4 + \ldots\right)\tfrac{1}{2}(\nabla\mathbf{u}_\infty + (\nabla\mathbf{u}_\infty)^{\mathrm{T}})\right\}^0.$$

When $a = b = c$, $D^2 = a^2\nabla^2$, and since $(\nabla^4\mathbf{u}_\infty)^0 = 0$ and $\mathbf{Z}$ becomes isotropic, we recover the result (3) for a sphere. The result above seems to be new.

3. The Faxén relations for an almost-spherical viscous drop

3.1. *A reciprocal theorem*

We now consider a drop of fluid of viscosity $\lambda\mu$ and surface S immersed in an unbounded fluid of viscosity μ with a prescribed flow $\mathbf{u}_\infty$ (and corresponding stress $\boldsymbol{\sigma}_\infty$) at a large distance from the drop. A body force $\nabla\psi$ acts on the drop such that it remains stationary, and a surface tension γ acts across S.

Then the fluid velocity satisfies

$$\left.\begin{aligned}\nabla.\mathbf{u} = 0, \quad \nabla.\boldsymbol{\sigma} = 0\\ \boldsymbol{\sigma} = -p\mathbf{I} + \mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})\end{aligned}\right\}\text{ outside the drop,}$$
$$\left.\begin{aligned}\nabla.\mathbf{u} = 0, \quad \nabla.\boldsymbol{\sigma} = \nabla\psi\\ \boldsymbol{\sigma} = -p\mathbf{I} + \lambda\mu(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}})\end{aligned}\right\}\text{ inside the drop}$$

with $\mathbf{u}\to\mathbf{u}_\infty$ as $r\to\infty$, $\mathbf{u}.\mathbf{n} = 0$ on S, $[u] = 0$ and $[\boldsymbol{\sigma}.\mathbf{n}] = \gamma\mathbf{n}$, where square brackets denote the jump across S. $\mathbf{u}'$, $\boldsymbol{\sigma}'$, $\mathbf{u}'_\infty$, γ' and ψ' satisfy a similar set of equations. Then we claim that

$$\int_{S_R} \mathbf{u}'.\boldsymbol{\sigma}.\mathbf{n}\,dS = \int_{S_R} \mathbf{u}.\boldsymbol{\sigma}'.\mathbf{n}\,dS,$$

where S_R is any surface which encloses the drop. This result is easy to prove. It follows that

$$\int_{S_R} (\mathbf{u}'_\infty . \boldsymbol{\sigma} . \mathbf{n} - \mathbf{u} . \boldsymbol{\sigma}'_\infty . \mathbf{n})\, dS = \int_{S_R} [(\mathbf{u}'_\infty - \mathbf{u}') . \boldsymbol{\sigma} . \mathbf{n} - \mathbf{u} . (\boldsymbol{\sigma}'_\infty - \boldsymbol{\sigma}') . \mathbf{n}]\, dS$$

and, with errors $O(l/R)$, $\boldsymbol{\sigma} . \mathbf{n}$ and $\mathbf{u}$ on the right-hand side may be replaced by $\boldsymbol{\sigma}_\infty . \mathbf{n}$ and $\mathbf{u}_\infty$. Thus by allowing R to become arbitrarily large we obtain

$$\int_{S_\infty} (\mathbf{u}'_\infty . \boldsymbol{\sigma} . \mathbf{n} - \mathbf{u} . \boldsymbol{\sigma}'_\infty . \mathbf{n})\, dS = \int_{S_\infty} [(\mathbf{u}'_\infty - \mathbf{u}') . \boldsymbol{\sigma}_\infty . \mathbf{n} - \mathbf{u}_\infty . (\boldsymbol{\sigma}'_\infty - \boldsymbol{\sigma}') . \mathbf{n}]\, dS. \tag{6}$$

3.2. *Force on the drop*

We first choose $\mathbf{u}'_\infty = \mathbf{U}'$ and $\boldsymbol{\sigma}'_\infty = 0$. Then as shown by Batchelor (1967, §4.9), the spherical shape of the drop is preserved even when $\gamma = 0$ and

$$\mathbf{u}'_\infty - \mathbf{u}' = \mathbf{U}' . (\mathbf{I} + \mathbf{rr}/r^2)\beta a/2r + \mathbf{U}' . (3\mathbf{rr}/r^2 - \mathbf{I})\tfrac{1}{4}(1-\beta)a^3/r^3$$

with

$$(\boldsymbol{\sigma}'_\infty - \boldsymbol{\sigma}') . \mathbf{r}/r = \frac{\mu}{a}\left\{\frac{3\mathbf{r} . \mathbf{U}'\mathbf{r}}{r^2}\left(-\beta\frac{a^2}{r^2} + 3(\beta - 1)\frac{a^4}{r^4}\right) + 3\mathbf{U}'(1-\beta)\frac{a^4}{r^4}\right\},$$

where $$\beta = (2 + 3\lambda)/(2 + 2\lambda).$$

Hence substituting in (6), and noting that $(\nabla^4\mathbf{u})^0 = 0$, gives directly

$$\mathbf{F} = 4\pi\mu a\beta\left(\mathbf{u}_\infty + \frac{\lambda}{2(2+3\lambda)}\nabla^2\mathbf{u}_\infty\right)^0.$$

This result was first obtained by Hetsroni & Haber (1970), by means of a full solution for the fluid velocity inside and outside the drop. In the case $\lambda = \infty$ we recover (1).

3.3. *The stresslet exerted by the drop*

We now choose $\mathbf{u}'_\infty = \mathbf{E}' . \mathbf{r}$ and $\boldsymbol{\sigma}'_\infty = 2\mu\mathbf{E}'$. Then as in Cox (1969), for sufficiently large values of $\gamma/\mu a$ the drop is nearly spherical and

$$\mathbf{u}'_\infty - \mathbf{u}' = 5\alpha_1 \mathbf{r} . \mathbf{E}' . \mathbf{rr}a^3/r^5 + \alpha_2(2\mathbf{E}' . \mathbf{r} - 5\mathbf{r} . \mathbf{E}' . \mathbf{rr}/r^2)a^5/r^5 \tag{7}$$

with

$$(\boldsymbol{\sigma}'_\infty - \boldsymbol{\sigma}') . \mathbf{r} = -2\mu[20\alpha_1 a^3\mathbf{r} . \mathbf{E}' . \mathbf{rr}/r^5 - 5\alpha_1 \mathbf{E}' . \mathbf{r}a^3/r^3 + 8\alpha_2 \mathbf{E}' . \mathbf{r}a^5/r^5 - 20\alpha_2 \mathbf{r} . \mathbf{E}' . \mathbf{rr}a^5/r^7], \tag{8}$$

where $$\alpha_1 = (5\lambda + 2)/[10(\lambda + 1)], \quad \alpha_2 = \lambda/2(\lambda + 1).$$

There are two contributions to these velocity and stress fields. The presence of a *spherical* drop with or without surface tension in the given flow induces an instantaneous velocity given by (7) and (8) but with $\alpha_1 = \alpha_2 = (\lambda - 1)/(2\lambda + 3)$. This term plainly vanishes when the viscosities of the two phases are the same. The second contribution arises from the product of the large surface tension of the drop and its small deviation from sphericity. The assumption that the drop shape is in equilibrium with the imposed flow implies that this term is comparable in magnitude with the one previously discussed. It generates a velocity field of the form of (7) and (8) with

$$\alpha_1 = (19\lambda + 16)/10(\lambda + 1)(2\lambda + 3)$$

and $$\alpha_2 = (3\lambda + 2)/2(\lambda + 1)(2\lambda + 3).$$

Now substituting these results into (6) gives

$$\mathbf{S} = \tfrac{4}{3}\pi\mu a^3 \frac{(5\lambda+2)}{2(\lambda+1)}\left\{\left(1+\frac{\lambda}{2(5\lambda+2)}\nabla^2\right)(\nabla\mathbf{u}_\infty+(\nabla\mathbf{u}_\infty)^{\mathrm{T}})\right\}^0.$$

This agrees with (3) in the limit $\lambda\to\infty$. For other values of λ this result seems not to have been written down before.

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